Mechanical wave in tube – theory
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Introduction

The quantitative description of mechanical wave propagation in tubes is remarkably important for biomechanics (e.g. analysis of pulse wave propagation in cardiovascular system) and also for industry (e.g. analysis of wave propagation in pipelines). Nevertheless, the comprehensive theory of mechanical wave propagation in tube is not currently at hand. The solution of problems connected with dynamic mechanical behavior of tubes requires firstly suggestion of an adequate mechanical model. Such a model must take into consideration continuously distributed (inertial, elastic and viscose) parameters in entire system. The crucial next step is the derivation of constitutive equations for mechanical wave propagation in the system. From the mechanical point of view, any elastic or viscoelastic tube filled with viscose liquid is highly complicated system. Mechanical behavior of such structures depends on theirs geometry and on inertial, elastic and viscose forces in wall as well as in liquid inside. Thus we face a considerably complicated task. Helpful may be the sophisticated methodology of derivation of telegraph equations for electric waves in cables applied by W Thompson more than hundred years ago, as the fundamental mechanical and electrical equations are analogical. Knowledge of constitutive equations of mechanical wave propagation in tubes enables calculations of speed of wave and damping of wave. Further, it enables calculation of wave impedance and reflections of waves. Also, the strains and stresses in tubes walls may be determined. The constitutive equations can be also used for the inverse problems solutions. Typically, the measurement of the propagation of pulse wave in cardiovascular system may be used for the estimation of viscoelastic parameters changes in arterial walls.

1. Premises, goals and principles

This analysis is concerned with mechanical waves in a cylindrical tube with thin elastic or viscoelastic wall filled with a viscose liquid of defined density. The tube is considered to be composed of an infinite number of segments (Fig. 1). The geometry and the corresponding symbols are in Figure 2. We will analyze the situation when forces (pressures) and deformations (displacements) are changing in time domain. The analysis aims to provide formulae for calculation of speed of wave propagation, damping of waves, waves mechanical impedance, reflections of waves and conditions for mechanical matching of tubes and tube endings.
2. **Force equilibrium in tube** (Fig. 3)
2.1. Forces in liquid inside tube (Fig. 4)

Inertial forces
As the wall of tube is thin, mass of wall is negligible with respect to mass of liquid inside. The geometry of segments is changing during deformation (Fig.4); the position of the center of inertia is thus also changing. Inertial forces obey Newton’s law of inertia.
\[ F_M = m \frac{dY_M}{d^2t} \]

If the liquid is homogenous, it holds

\[ F_M = \rho V \frac{dY_M}{d^2t} \]

and also

\[ F_M = \rho \pi Y^2 \frac{dY_M}{d^2t} X \]

where \( Y_M \) is the position of the center of inertia (radius), \( m \) is the mass of segment, \( \rho \) is the density, \( V \) is the volume of segment.

If the tube is circular, it holds

\[ Y_M = \frac{2}{3} Y \quad F_M = \frac{2}{3} \pi \rho Y^2 \frac{dY}{d^2t} X \quad M = \frac{2}{3} \pi \rho Y^2 \frac{dy^2}{d^2t} X \]

In case of unit length tube, the inertial force \( F_m \) is

\[ F_m = \frac{2}{3} \pi \rho \frac{dy^2}{d^2t} \]

The relationships between forces and deformations in rheological models (see e.g. Fig. 10) are as follows

\[ F_m = M \frac{dy^2}{d^2t} \]

where \( M \) is the inertial coefficient.

In case of cylindrical tube of unit geometry, for the inertial coefficient \( M \) thus it holds

\[ M = \frac{2}{3} \pi \rho Y^2 \]

Radial viscose forces (Fig. 5)

The geometry of segments is changing during deformation. The relative movement of layers in liquid determines radial viscose forces. The tangential stress obeys Newton’s law of viscosity.

\[ \tau = \eta \frac{dv}{dz} \]

where \( \eta \) is the viscosity, \( v \) is the speed of deformation of layer, \( z \) is the distance of layers.
The speed of outer layer (on perimeter of cylinder) is
\[ v = 2\pi \frac{dY}{dt} \]
The speed of layer in the center (in the axis of cylinder) is zero.
The speed gradient is
\[ \frac{dy_L}{dz} = 2\pi \frac{dY}{dt} \]
where \( y_L \) is the speed of layer.
The gradient is the same for any position of layers (Fig. 6).
Viscose forces $F_V$ straining layers

$$F_v = A \eta \frac{dy}{dz}, \quad F_v = z X \eta 2 \pi \frac{dY}{dt}$$

where $A$ is the surface of layer.

The position of center of viscose forces in cylinder is

$$Y_M = \frac{2}{3} Y$$

The viscose forces in segment are thus

$$F_v = \frac{2}{3} \eta 2 \pi X \frac{dY}{dt}, \quad F_v = \frac{4}{3} \eta X \frac{dY}{dt}, \quad F_v = \frac{4}{3} \eta X \frac{dy}{dt}$$

The viscose force $F_v$ in cylindrical segment of unit length is

$$F_v = \frac{4}{3} \eta \frac{dy}{dt} \quad (3)$$

The equation for relationships between forces and deformations in rheological models (see e.g. Fig. 9) is as follows

$$F = N \frac{dy}{dt}$$

where $N$ is the Newton (viscose) coefficient.

For the Newton coefficient $N$ thus it holds, in case of unit cylindrical segment

$$N = \frac{4}{3} \eta \quad (4)$$

Axial viscose forces

The influx and the outflow of liquid determine axial viscose forces. In short segment these forces are negligible.

2.2. Forces in wall (see Fig. 7 an Fig. 8)
Fig. 7. Geometry of deformation of wall - detail

Geometry of deformation of wall - detail

- $D$: thickness of wall
- $X$: length of segment
- $F_T$: shear force
- $y$: deformation

Diagram shows a cross-section of a wall with labeled dimensions and forces.
Shear force in deformed segment

\[
\text{shear forces}
\]

\[
D \text{ thickness of wall}
\]

\[
s = 2 \pi Y
\]

\[
\text{unrolled wall}
\]

Fig. 8. Shear force in deformed segment

Elastic shear stress in the wall

\[
\tau_e = G \frac{Y}{X}
\]

\[
\frac{F_{TE}}{A_Y} = G \frac{Y}{X}
\]

\[
\frac{F_{TE}}{2 \pi Y D} = G \frac{Y}{X}
\]

where \(G\) is the modulus of shear stress, \(A\) is the area of wall cross section.

Elastic tangent force in the wall

\[
F_{TE} = G \ 2 \pi Y \ D \ \frac{y}{X}
\]

Elastic tangent force in the wall of unit segment

\[
F_{TE} = G \ 2 \pi Y \ D \ y
\]  
(5)
Hooke coefficient in the wall of unit segment

\[
H = G 2\pi Y D
\]  \hspace{1cm} (6)

Viscose tangent stress

\[
\tau_v = \eta_p \frac{dy}{dt} \frac{1}{X} \quad \frac{F_{TV}}{A_T} = \eta_p \frac{dy}{dt} \frac{1}{X} \quad \frac{F_{TV}}{2\pi Y D} = \eta_p \frac{dy}{dt} \frac{1}{X}
\]

where \( \eta_p \) is the viscosity of the wall.

Viscose tangent force

\[
F_{TV} = \eta_p \ 2\pi Y D \frac{dy}{dt} \frac{1}{X}
\]

The viscose tangent force in unit segment

\[
F_{TV} = \eta_p \ 2\pi Y D \frac{dy}{dt}
\]  \hspace{1cm} (7)

Newton coefficient \( N_p \) in unit segment

\[
N_p = \eta_p \ 2\pi Y D
\]  \hspace{1cm} (8)

3. Constitutive wave equations for Maxwell model of the wall

3.1. Rheological scheme of tube for Maxwell model of wall

Fig.9. Rheological scheme for Maxwell model of wall
**Applied formulae**

For simplicity, we use the speed of deformation \((v)\) instead the deformation \((y)\) in following formulae.

**Inertial force in liquid**

\[
F_m = \frac{2\pi \rho Y^2}{3} \frac{dv}{dt}
\]

**Viscose force in liquid**

\[
F_v = \frac{4\pi}{3} \eta v
\]

**Derivation of elastic force in wall**

\[
\frac{dF_{TE}}{dt} = G \ 2\pi Y D v
\]

**Viscose force in wall**

\[
F_{TV} = \eta_p \ 2\pi Y D v
\]

**Derivation of constitutive equations**

The movement of liquid leads to decline of force on wall as follows

\[
F - (F + \frac{\partial F}{\partial x} dx) = M \ dx \ \frac{\partial v}{\partial t} + N \ dx \ v
\]

Thus it holds

\[
\frac{\partial F}{\partial x} = M \ \frac{\partial v}{\partial t} + N \ v \quad (9)
\]

The movement of wall leads to decline of speed of deformation

\[
v - (v + \frac{\partial v}{\partial x} dx) = \frac{1}{H} dx \ \frac{\partial F}{\partial t} + \frac{1}{N} dx \ F
\]

Thus it holds

\[
\frac{\partial v}{\partial x} = \frac{1}{H} \frac{\partial F}{\partial t} + \frac{1}{N_p} F \quad (10)
\]

### 3.2. Wave equations

From equations (9) and (10) the constitutive (wave) equations follows

\[
\frac{\partial^2 F}{\partial x^2} = M \ \frac{\partial^2 F}{\partial t^2} + \left( \frac{N}{H} + \frac{M}{N_p} \right) \frac{\partial F}{\partial t} \quad (11)
\]
\[ \frac{\partial^2 v}{\partial x^2} = \frac{M}{H} \frac{\partial^2 v}{\partial t^2} + \left( \frac{N}{H} + \frac{M}{N_p} \right) \frac{\partial v}{\partial t} \]  

(12)

3.3. Speed of wave propagation

The solution of equations (11) and (12) leads to the formula for calculation of speed of wave

\[ c = \frac{\omega}{\alpha} \]  

(13)

where

\[ \alpha = \sqrt{\frac{1}{2} \left( \frac{N}{N_p} + \omega^2 \frac{M}{H} \right) + \frac{1}{2} \left( \frac{1}{N_p^2} + \omega^2 \frac{1}{H^2} \right)} \]  

(14)

where

\[ N = \frac{4 \pi}{3} \eta \quad N_p = \eta_p 2 \pi Y D \quad H = G 2 \pi Y D \quad M = \frac{2 \pi \rho}{3} Y^2 \]

Example 1 – Speed of wave propagation on frequency

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Tube radius (Y mm)</th>
<th>Wall thickness (D mm)</th>
<th>Wall modulus (G MPa)</th>
<th>Liquid density (( \rho ) kg/l)</th>
<th>Liquid viscosity (( \eta ) Pa s)</th>
<th>Wall viscosity (( \eta_p ) Pa s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable</td>
<td>4.5</td>
<td>0.8</td>
<td>0.3</td>
<td>1</td>
<td>0.003</td>
<td>300</td>
</tr>
</tbody>
</table>

Tab. E1. The variable and parameters of tube

Fig. E1. Dependency of speed of wave propagation on frequency for viscoelastic tube
**Example 2 – Speed of wave propagation on radius**

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Tube radius</th>
<th>Wall thickness</th>
<th>Wall modulus</th>
<th>Liquid density</th>
<th>Liquid viscosity</th>
<th>Wall viscosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$ (Hz)</td>
<td>$Y$ (mm)</td>
<td>$D$ (mm)</td>
<td>$G$ (MPa)</td>
<td>$\rho$ (kg/l)</td>
<td>$\eta$ (Pa s)</td>
<td>$\eta_p$ (Pa s)</td>
</tr>
<tr>
<td>1</td>
<td>variable</td>
<td>0.8</td>
<td>0.3</td>
<td>1</td>
<td>0.003</td>
<td>300</td>
</tr>
</tbody>
</table>

Tab. E2. The variable and parameters of tube

![Graph](image)

Fig. E2. Dependency of speed of wave propagation on wall radius for viscoelastic tube

**Example 3 – Speed of wave propagation on wall thickness**

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Tube radius</th>
<th>Wall thickness</th>
<th>Wall modulus</th>
<th>Liquid density</th>
<th>Liquid viscosity</th>
<th>Wall viscosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$ (Hz)</td>
<td>$Y$ (mm)</td>
<td>$D$ (mm)</td>
<td>$G$ (MPa)</td>
<td>$\rho$ (kg/l)</td>
<td>$\eta$ (Pa s)</td>
<td>$\eta_p$ (Pa s)</td>
</tr>
<tr>
<td>1</td>
<td>4.5</td>
<td>variable</td>
<td>0.3</td>
<td>1</td>
<td>0.003</td>
<td>300</td>
</tr>
</tbody>
</table>

Tab. E3. The variable and parameters of tube
Fig. E3. Dependency of speed of wave propagation on wall thickness for viscoelastic tube

**Example 4 – Speed of wave propagation on wall shear modulus**

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Tube radius</th>
<th>Wall thickness</th>
<th>Wall modulus</th>
<th>Liquid density</th>
<th>Liquid viscosity</th>
<th>Wall viscosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$ (Hz)</td>
<td>$Y$ (mm)</td>
<td>$D$ (mm)</td>
<td>$G$ (MPa)</td>
<td>$\rho$ (kg/l)</td>
<td>$\eta$ (Pa.s)</td>
<td>$\eta_p$ (Pa.s)</td>
</tr>
<tr>
<td>1</td>
<td>4.5</td>
<td>0.8</td>
<td>variable</td>
<td>1</td>
<td>0.003</td>
<td>300</td>
</tr>
</tbody>
</table>

Tab. E4. The variable and parameters of tube

Fig. E4. Dependency of speed of wave propagation on shear modulus of wall for viscoelastic tube
3.4. Damping of wave

\[ DP = e^{\beta x} \]  

(15)

where

\[ \beta = \sqrt{\frac{1}{2} \left( \frac{N}{N_p} - \omega^2 \frac{M}{H} \right) + \frac{1}{2} \left( \frac{1}{N_p^2} + \omega^2 \frac{1}{H^2} \right) (N^2 + \omega^2 M^2)} \]  

(16)

Example 5 – Damping of wave on tube length

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Tube radius</th>
<th>Wall thickness</th>
<th>Wall modulus</th>
<th>Liquid density</th>
<th>Liquid viscosity</th>
<th>Wall viscosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>f (Hz)</td>
<td>Y (mm)</td>
<td>D (mm)</td>
<td>G (MPa)</td>
<td>( \rho ) (kg/l)</td>
<td>( \eta ) (Pa s)</td>
<td>( \eta_p ) (Pa s)</td>
</tr>
<tr>
<td>1</td>
<td>4.5</td>
<td>0.8</td>
<td>0.3</td>
<td>1</td>
<td>0.003</td>
<td>300</td>
</tr>
</tbody>
</table>

Tab. E5. Parameters of tube

Fig. E5. Dependency of damping on tube length for viscoelastic tube

3.5. Wave mechanical impedance

\[ Z_0 = \sqrt{\frac{1}{N_p} + i \omega \frac{1}{H} - \frac{1}{N + i \omega M}} \]  

(17)
3.6. Wavefront of non-harmonic wave

Harmonic analysis

The equations (13 - 17) hold for simple harmonic (sinusoidal) waves. The harmonic analysis enables extend theirs applicability on all periodic functions.

Methodology of harmonic analysis consists in the representation of periodic functions as the superposition of harmonic functions (harmonics). As superposition principle holds for solutions of a linear homogeneous ordinary differential equation, any periodic function may be approximate as a Fourier series. A Fourier series is an approximation of a periodic function in terms of an infinite sum of sines and cosines. Then it is possible to recombine harmonics to obtain the solution to the original problem.

Fourier series

According definition, for the Fourier series the following holds

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{2\pi n}{T} x \right) + b_n \sin \left( \frac{2\pi n}{T} x \right) \right] \]  

(18)

where \( f(x) \) is the periodic function, \( T \) is the period, \( n \) is the integer.

For coefficients \( a \) resp. \( b \) in equation (18) it holds:

\[ a_0 = \frac{1}{T} \int_0^T f(\tau) \, d\tau \]  

(19)

\[ a_n = \frac{2}{T} \int_0^T f(\tau) \cos \left( \frac{2\pi n}{T} \tau \right) \, d\tau \]  

(20)

\[ b_n = \frac{2}{T} \int_0^T f(\tau) \sin \left( \frac{2\pi n}{T} \tau \right) \, d\tau \]  

(21)

Thus in time domain it holds:

\[ f(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos (n \omega t) + b_n \sin (n \omega t) \right] \]  

(22)

where \( t \) is the time, \( \omega \) is the angular speed.

\[ a_n = \frac{2}{T} \int_0^T f(\tau) \cos (n \omega \tau) \, d\tau \]  

(23)

\[ b_n = \frac{2}{T} \int_0^T f(\tau) \sin (n \omega \tau) \, d\tau \]  

(24)
**Example 6 – Propagation of non-harmonic wave**

Suppose the viscoelastic tube of following parameters:

<table>
<thead>
<tr>
<th>Tube radius</th>
<th>Wall thickness</th>
<th>Wall modulus</th>
<th>Liquid density</th>
<th>Liquid viscosity</th>
<th>Wall viscosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$ (mm)</td>
<td>$D$ (mm)</td>
<td>$G$ (MPa)</td>
<td>$\rho$ (kg/l)</td>
<td>$\eta$ (Pa s)</td>
<td>$\eta_p$ (Pa s)</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0.3</td>
<td>1</td>
<td>0.003</td>
<td>300</td>
</tr>
</tbody>
</table>

Tab. E6. Parameters of tube

Suppose further that wavefront of pulse wave at point A (see Fig. E6) is according Fig. E7.

![Fig. E6. Propagation of wave in tube](image)

The curve in Figure E7 is the sum of 3 harmonics.

In general, for harmonics it holds: $L_n = A_n \sin (n \pi f t + \varphi_n)$ where $f$ is the frequency of the first harmonic.

For the curve in Figure E7 it holds

$A_1 = 0.25, \quad f = 1 \text{ Hz}, \quad \varphi_1 = 0$

$L_1 = 0.25 \sin (2 \pi t)$

For the second harmonic it holds:

$A_2 = 0.25, \quad \varphi_2 = 1.5 \pi$

$L_2 = 0.25 \sin (4 \pi t + 1.5 \pi)$

For the third harmonic it holds:

$A_3 = 0.5, \quad \varphi_3 = \pi$
\[ L_2 = 0.5 \sin(6\pi t + \pi) \]

Wavefront of pulse wave at point A \((x = 0)\) is the sum of the harmonics at point A (see Fig. E7)

According formulae (13 and 14) is the speed of wave propagation \(c \approx 10 \text{m/s}\) for all harmonics. The phase shifts of harmonics at point B are thus possible to calculate.

According formulae (15 and 16), damping of the harmonics at point B may be calculated.

Consequently, parameters of harmonics at point B are possible to obtain.

Wavefront of pulse wave at point B \((x = 0.5 \text{ m})\) is the sum of the harmonics at point B (see Fig. E8)

![Fig. E7. Wavefront at point A](image1)

![Fig. E8. Wavefront at point B](image2)

The comparison of wavefronts at A and at B is shown in Figure E9.

![Fig. E9. Comparison of wavefronts (full line wavefront at A, dotted line at B)](image3)

*Comment: In reality, for calculation computer with adequate software is necessary to use.*
4. Constitutive wave equations for Voigt model of wall

4.1. Rheological scheme of tube for Voigt model of wall

![Rheological scheme for Voigt model of wall](image)

Fig.10. Rheological scheme for Voigt model of wall

4.2. Transformation of Voigt model to Maxwell model

In many situations, rheological parameters \((H_V, N_{PV})\) of wall are available for Voigt model (e.g. if measurement is based on torsion oscillations). In this case, the Voigt model is appropriate. Nevertheless, the constitutive equations derived for Maxwell model are also applicable, provided rheological parameters of Voigt model are transformed into Maxwell ones \((H, N_P)\). The transforming equations are as follows:

\[
H \cong H_V \\
N_P \cong k^2 N_{PV} \\
k = \frac{H_V}{N_{PV}}
\]

The previous formulae are applicable if this holds: \(k \gg 10\), which is usually fulfilled.
**Literature:**

Doubal S, Klemera P.: Pulse Wave in Viscoelastic Tube. Folia Pharmaceutica University Caroline XXXIII, pp. 95 – 99, 2005


